Final exercise set for QFT1

January 13, 2020

1 Exercise 1: Gordon identities

Consider u and v, the positive and negative frequency solutions of the Dirac equation for a particle of mass m in momentum space, and two tetramomenta p and k for on-shell particles, i.e. $p^2 = k^2 = m^2$. Derive the Gordon identities for u [3pt] and v [3pt]:

$$\bar{u}(p)\gamma^\mu u(q) = \bar{u}(p)\left[\frac{(p+q)^\mu}{2m} + \frac{2i\sigma^{\mu\nu}(p-q)_\nu}{2m}\right]u(q) \tag{1}$$

$$\bar{v}(p)\gamma^\mu v(q) = -\bar{v}(p)\left[\frac{(p+q)^\mu}{2m} + \frac{2i\sigma^{\mu\nu}(p-q)_\nu}{2m}\right]v(q) \tag{2}$$

where γ^μ are matrices satisfying the Clifford algebra and $\sigma^{\mu\nu} = i\left[\gamma^\mu, \gamma^\nu\right]/4$.

2 Exercise 2: free vectorial massive field

Starting from a real vector field A^μ, the Proca density of Lagrangian is obtained by adding to the Maxwell Lagrangian a mass term:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{3}$$

with $m > 0$. Discuss the quantization of such a theory. This exercise can be solved in different ways and the student can choose one of them (e.g. the canonical quantization, as done in the lectures for the electromagnetic field, or the particle quantization, as done for the scalar field). It is important to discuss:

- the gauge invariance of the Lagrangian: is this Lagrangian invariant under $A_\mu \to A_\mu + \partial_\mu f$? [1pt];
- the equations of motion and their solutions [4pt]; hint: compute the Euler-Lagrange equations from the density of Lagrangian. Apply ∂_ν to the equation for A^ν to get an additional condition for $\partial_\nu A^\nu$;
- the physical degrees of freedom of the quantized particles [3pt];
- the commutation (or anticommutation) relation in spacetime and the Feynman propagator [4pt].

General hint: A closer analogy to the massless case can be misleading. Try not to modify the Lagrangian and not to use an approach similar to the Gupta-Bleuler one. In particular, try to quantize only the relevant degrees of freedom.

3 Exercise 3: decay of a massive scalar particle

Consider two types of neutral, scalar particles ξ and η of masses $0 < m_\xi < m_\eta$. The two particles are described by two scalar, Hermitian fields $\hat{\phi}$ and $\hat{\Phi}$ with a density of Lagrangian:

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\hat{\phi}\,\partial^\mu\hat{\phi} - \frac{1}{2}m_\xi^2\hat{\phi}^2 + \frac{1}{2}\,\partial_\mu\hat{\Phi}\,\partial^\mu\hat{\Phi} - \frac{1}{2}m_\eta^2\hat{\Phi}^2 - \mu\,\hat{\Phi}\hat{\phi}^2\,. \tag{4}$$

This lagrangian allows the decay of a particle η into ξ particles.

- Find the dimensions (in natural units) of the coupling constant μ and $n_{\min}$, the minimum amount of ξ particles in the final state [2pt];
- find the lowest order contribution in μ to the matrix element of the transition matrix $i\hat{T}$ for the decay of a η particle into $n_{\min}$ ξ particles [4pt]; suggestion: remembering the LSZ formula, if you want, you can drop the $(2\pi)^{9/2}$ factor that comes out from the direct calculations.
- find the conditions on the particle masses such that the decay in $n_{\min}$ particles is kinematically allowed [1pt];
- under these conditions, compute the lifetime of a η particle at lowest order in μ [4pt].

The lifetime is defined as the inverse of the decay rate Γ. The generic formula for the differential decay rate for one particle η decaying into n ξ particles is very similar to the one of the differential cross section:

$$\mathrm{d}\Gamma = \frac{S}{2m_\eta}\,(\mathrm{d}\Pi_n)\,|\mathcal{M}(\xi \to \eta_1, \ldots, \eta_n)|^2 \tag{5}$$

where $(\mathrm{d}\Pi_n)$ is the invariant n-particle final state:

$$(\mathrm{d}\Pi_n) = \prod_{i=1,n} \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i}\,(2\pi)^4\,\delta_{(4)}\left(p_\xi - \sum_{i=1,n} p_i\right) \tag{6}$$

while S is the symmetry factor for identical particles in the final state:

$$S = \prod_{j=1,m}\frac{1}{n_j!} \tag{7}$$

where the index j runs over all m kinds of different particles in the final state and n_j is the number of identical particles j in the final state, such that: $\sum_{j=1,m} n_j = n$.

4 Exercise 4: positron-positron scattering

Compute the unpolarized, differential cross-section in the center of mass frame for the scattering of two positrons into two positrons at leading order in QED:

$$e^+ + e^+ \to e^+ + e^+\,. \tag{8}$$

In particular:

- find the initial and final particle states, draw the relevant Feynman diagrams, and find their amplitudes [4pt];
- compute the unpolarized squared matrix element and express it in terms of the Mandelstam variables [4pt];
- calculate the differential cross section in the center of mass frame [3pt];
- discuss both the non-relativistic and the ultra-relativistic limits [2pt].

5 Notes on the solutions

- the solutions must be in English;
- try to include all and only the relevant points in your answer;
- 2/3 of the total points are necessary to access to the oral exam.

6 Possible references

- S. Weinberg, Quantum Field Theory, vol 1 [Wei]
- F. Mandl & G. Shaw, Quantum Field Theory [Man]
- M. Peskin & D. Schroeder, An introduction to Quantum Field Theory [Pes]
- M. Maggiore, A Modern Introduction to Quantum Field Theory [Mag]
- J. Sakurai, Modern Quantum Mechanics [Sak]

7 List of topics and possible detailed references

In the following, you can find a rough index of the topics covered in the lectures and some related references. These paragraphs and chapters from books often contain more material than the ones covered in the lectures. I am reporting them here only in case you need additional material to clarify the lectures content. For the exam, it is not necessary to study topics and concepts not covered in the lectures. Of course, you can study them for your pleasure or for future convenience!

- **recap of QM and pictures for time evolution**
  Of course, any good book in QM, e.g. chapters 1 and 2 of [Sak]. A good summary: [Wei] paragraph 2.1; appendix 1.5 of [Mas].
- **Lorentz and Poincaré groups**
  Chapter 2 of [Mag]; paragraphs 2.2-2.3-2.4-2.5 [Wei]. For the rotation group, see also capter 3 of [Sak].
- **Classical field theory and symmetries in classical mechanics**
  Paragraph 3.1 and 3.2 of [Mag]; Capter 2 of [Man], paragraph 2.2 of [Pes].
- **The (Hermitian and non-Hermitian) free scalar field**
  Paragraph 3.3 and 4.1 of [Mag]; Chapter 3 of [Man]; paragraphs 2.3 and 2.4 of [Pes].
- **The Dirac equation and the free Dirac field**
  Paragraph 3.4 and 4.2 of [Mag]; Chapter 4 of [Man]; chapter 3 of [Pes].
- **The electromagnetic field**
  Paragraph 3.5 and 4.3.2 of [Mag]; Chapter 5 of [Man].
- **Diffusion theory: in and out states and S matrix**
  Paragraph 3.1, 3.2 and 3.3 of [Wei]; paragraph 5.1 and 5.3 of [Mag]; chapter 6 of [Man]; paragraphs 4.1-4.2-4.3-4.4 of [Pes].
- **ϕ-4 theory and LSZ formula**
  Paragraph 5.2 and 5.5 of [Mag]; paragraphs 4.2-4.3-4.4-4.6 of [Pes].
- **Cross section and 2 body phase space**
  Paragraphs 4.5 of [Pes] and paragraph 6.4 of [Mag].
- **Quantum electrodynamics**
  Paragraphs 4.7-4.8-5.1-5.4 of [Pes]. Chapter 7 and 8 of [Man].