1 Exercise 1: particle decay in ϕ^4 theory

Let $\hat{\phi}(x)$ be a quantized Hermitian scalar field describing a scalar particle χ of mass $m \neq 0$ and following a Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{2}\, m^2 \hat{\phi}^2 - \frac{\lambda}{4!}\, \hat{\phi}^4 \, .$$

Is it possible for a particle χ to spontaneously decay? Which is the minimum amount of decay products? Discuss your answer both in terms of dynamics (i.e., find the lowest order Feynman diagram and compute its non-vanishing amplitude using e.g. Feynman rules) and kinematics (i.e., is this process allowed in nature?).

2 Exercise 2: Relativistically invariant two-particles phase space

Show that for a two particles final state the relativistically invariant two-particles phase space is

$$\int \mathrm{d}\Pi_2 = \int \mathrm{d}\Omega \frac{1}{16\pi^2} \frac{|\mathbf{p}_{\mathrm{CoM}}|}{E_{\mathrm{CoM}}}$$

where $|\mathbf{p}_{\mathrm{CoM}}|$ refers to the modulus of the impulse of each of the two particles in the center of mass frame, while $E_{\mathrm{CoM}}$is the total energy in the same frame. Use this formula to show that the differential cross-section for a $2 \to 2$ process of four identical particles becomes:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CoM}} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{CoM}}^2} \, .$$

This formula is very relevant because it applies to any $2 \to 2$ process in the ultra-relativistic limit ($m \to 0$) for which there is symmetry for rotation along the collision axis .

3 Exercise 3: gamma matrix identities

Let γ^μ be 4 4x4 matrices that satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\, \mathbb{1}$$

where $\eta^{\mu\nu}$ is the flat space-time metric, and let Tr() denote the trace of a matrix. Prove the following identities:

- $\mathrm{Tr}\left(\gamma^\mu \gamma^\nu \gamma^\rho \ldots\right) = \mathrm{Tr}\left(\ldots \gamma^\rho \gamma^\nu \gamma^\mu\right)$
- $\gamma^\mu \gamma_\mu = 4\, \mathbb{1}$
- $\gamma^\mu \gamma^\nu \gamma_\mu = -2\, \gamma^\nu$
- $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\, \eta^{\nu\rho}\, \mathbb{1}$
- $\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\, \gamma^\sigma \gamma^\rho \gamma^\nu$