1 Exercise 1: Microcausality of charge-current density operator

Consider a quantized free Dirac field $\hat{\psi}(x)$ and its associated charge-current density operator:

$$\hat{j}^{\mu}(x) = -e\ \hat{\bar{\psi}}(x)\gamma^{\mu}\hat{\psi}(x) \tag{1}$$

Show that

$$[\hat{j}^{\mu}(x), \hat{j}^{\nu}(x)] = \mathbb{O} \tag{2}$$

if $(x-y)$ is space-like, i.e. $(x-y)^2 < 0$.

2 Exercise 2: Scalar field and anticommutators

Consider a free Hermitian scalar field $\hat{\phi}$. Show that if one postulates anticommutation relations:

$$\left\{a(\mathbf{k}), a^{\dagger}(\mathbf{p})\right\} = \delta_3\left(\mathbf{k}-\mathbf{p}\right)$$

and

$$\left\{a(\mathbf{k}), a(\mathbf{p})\right\} = \left\{a^{\dagger}(\mathbf{k}), a^{\dagger}(\mathbf{p})\right\} = 0$$

then

$$[\phi(x), \phi(y)] \neq 0$$

and

$$\{\phi(x), \phi(y)\} \neq 0$$

for space-like intervals, i.e. $(x-y)^2 < 0$. Discuss briefly the implication of this result. Hint: The commutator will result in an operator expression. To evaluate it, consider the expectation value on a single particle state or between the vacuum and a two particle states (in order to obtain a scalar quantity).

3 Exercise 3: Scalar QED

Consider a complex scalar field $\hat{\phi}$ (describing a particle χ and antiparticle $\bar{\chi}$) and a vector, Hermitian massless field $\hat{A}^{\mu}$ (describing photons). Consider the Lagrangian of the free Hermitian field and construct the intaracting Lagrangian by applying the minimal coupling prescription, i.e.

$$\partial_{\mu} \rightarrow D_{\mu} = \partial_{\mu} + ie\hat{A}_{\mu}$$

Which are the gauge transformations associated to the two fields, such that the action obtained from that Lagrangian is invariant? What happens if $\hat{\phi}$ is Hermitian? Hint: for the $\hat{A}^{\mu}$ field, consider the free Lagrangian:

$$-\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}.$$