Exercise set 5 for QFT1

November 15, 2019

1 Dirac eq solutions in Weyl representation

During the lecture we have solved Dirac's momentum equation for free, spin 1/2 particles in the standard/Dirac representation of the γ matrices. Obviosuly, this is not the only possible choice. The goal of this exercise is to find the solution using Weyl's representation. Show that these solutions can be written as

$$u^s(\mathbf{p}) = \begin{pmatrix} \sqrt{E_p \mathbb{1} - \mathbf{p}\cdot\sigma}\,\xi^s \\ \sqrt{E_p \mathbb{1} + \mathbf{p}\cdot\sigma}\,\xi^s \end{pmatrix} \qquad v^s(\mathbf{p}) = \begin{pmatrix} \sqrt{E_p \mathbb{1} - \mathbf{p}\cdot\sigma}\,\eta^s \\ -\sqrt{E_p \mathbb{1} + \mathbf{p}\cdot\sigma}\,\eta^s \end{pmatrix}$$

where it is understood that:

$$\xi^1 = \eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and the square root of an operator is obtained by appling the square root to the eigenvalue of the eigenstate. Just to be clear, if $\mathbf{p} = p\,\hat{z}$,

$$u^s(\mathbf{p}) = \begin{pmatrix} \left(\sqrt{E_p + p}\left(\frac{\mathbb{1}-\sigma_3}{2}\right) + \sqrt{E_p - p}\left(\frac{\mathbb{1}+\sigma_3}{2}\right)\right)\xi^s \\ \left(\sqrt{E_p + p}\left(\frac{\mathbb{1}+\sigma_3}{2}\right) + \sqrt{E_p - p}\left(\frac{\mathbb{1}-\sigma_3}{2}\right)\right)\xi^s \end{pmatrix}$$

Finally, compute the shape of the spinors $u^{1,2}$ and $v^{1,2}$ in the ultra-relativistic case (i.e., $m = 0$ case). A possible solution strategy is the following: assuming $m \neq 0$, solve first the problem in the particle rest frame. Then perform a generic boost along z, using the spinor representation of the Lorentz group.

2 Weyl field theory

In a previous exercise sheet, we have shown that, if ψ_R and ψ_L are 2-dimesional right-handed and left-handed Weyl spinors, respectively, then $\psi_R^\dagger \sigma^\mu \psi_R$ and $\psi_L^\dagger \bar{\sigma}^\mu \psi_L$ are tetra-vectors under Lorentz transformations. We can then introduce two lagrangian densities:

$$\mathcal{L}_R = i\psi_R^\dagger \sigma^\mu\,\partial_\mu \psi_R \qquad \mathcal{L}_L = i\psi_L^\dagger \bar{\sigma}^\mu\,\partial_\mu \psi_L\,.$$

- Compute the equation of motion for each of them;
- show that each component of ψ_R and ψ_L satisfies a massless Klein-Gordon equation;

- making the ansatz, $\psi_i = u_i e^{-ipx}$ with $i = L, R$ for the positive energy states, show that a left-handed massless Weyl spinor has helicity -1/2 while a right-handed massless Weyl spinor +1/2;
- if we consider at the same time a left and a right handed spinors, show that $\psi_L^\dagger \psi_R$ and $\psi_R^\dagger \psi_L$ are Lorentz scalars;
- consider the new Lagrangian density

$$\mathcal{L}_D = \mathcal{L}_R + \mathcal{L}_L - m \left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \right)$$

  and compute the (two) independent equations of motion;
- prove that ψ_L and ψ_R now satify the massive Klein-Gordon equation;
- by introducing the Dirac spinor as

$$\psi_D = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

  show that $\mathcal{L}_D$ is equivalent to the Dirac lagrangian introduced in the lecture, and the equations of motion reduces to the Dirac equation;
- finally, show that $u_{L,R}$ are no longer helicity eigenstate. What is the cause of that?

3 Chiral transformations

Let ψ be a Dirac field and consider the axial-vector current

$$j^{5,\mu} = \bar{\psi} \gamma^5 \gamma^\mu \psi$$

Prove that $j^{5,\mu}$ is conserved if and only if the field is massless (i.e. $m = 0$). Which is the symmetry transformation of ψ associated to this current? This transformation is said chiral transformation and it is a symmetry only for massless fields. Bonus: if you really like Weyl fields, how can one express the chiral transformation acting on ψ in terms of its ψ_L and ψ_R?