Exercise set 4 for QFT1

October 27, 2019

1 Heisenberg equation for the free Hermitian scalar field

Consider a free Hermitian scalar field $\hat{\phi}(x)$ in the Heisenberg picture. Show that the field satisfy the equation:

$$i\,\partial^\mu \hat{\phi}(x) = \left[\hat{\phi}(x), \hat{P}^\mu\right] .$$

The $\mu = 0$ equation is the Heisenberg equation for the field.

2 Plane wave expression for the free Hermitian field: an alternative approach

There are many different approaches to study the free Hermitian field and to obtain its plane wave expression. This is one of those. A free scalar field in space-time can be thought as a system of coupled harmonic oscillators. To show that, start from the Klein-Gordon Lagrangian. Derive the expression for the Hamiltonian:

$$\hat{H} = \int \mathrm{d}^3x\, \hat{\mathcal{H}}$$

where

$$\hat{\mathcal{H}} = \frac{1}{2}\left(\left(\partial^0\hat{\phi}\right)^2 + \left|\nabla\hat{\phi}\right|^2 + m^2\hat{\phi}^2\right)$$

Assume to decompose the field at a certain time t, $\hat{\phi}(t, \mathbf{x})$, in its Fourier modes:

$$\hat{\phi}(t, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}}\, e^{i\mathbf{k}\dot{\mathbf{x}}}\hat{q}(t, \mathbf{k})\,.$$

with $\hat{q}^\dagger(t, \mathbf{k}) = \hat{q}(t, -\mathbf{k})$ (why?). Show that $q(t, \mathbf{k})$ satisfies the equation:

$$\left(\partial_t^2 + \omega_{\mathbf{k}}^2\right) q(t, \mathbf{k}) = 0 \tag{1}$$

with $\omega_{\mathbf{k}}^2 = m^2 + \|\mathbf{k}\|^2$. Insert this expansion in the Hamiltonian and find explicitly that H is the sum of infinite harmonic oscillators in the Fourier space. Write the most general solution of eq. 1 in terms of $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$, insert it in the field Fourier expansion to find the field expression in Heisenberg picture.

3 Field expression for the momentum operator

Let $\hat{\phi}(x)$ be a free Hermitian operator in the Heisenberg picture. From the expression of the Klein-Gordon Lagrangian and of the energy-momentum tensor, prove first that the density of impulse is:

$$\hat{\mathcal{P}}^i = : \dot{\hat{\phi}}\, \partial^i \hat{\phi} : \ .$$

Prove then that the momentum operator defined as

$$\hat{P}^i = \int \mathrm{d}^3 x \, \hat{\mathcal{P}}^i$$

is equivalent to the one obtained in the lecture:

$$\hat{P}^i = \int \mathrm{d}^3 p \; p^i \, \hat{a}^\dagger(\mathbf{p}) \hat{a}(\mathbf{p})$$