Exercise set 3 for QFT1

November 5, 2019

1 The vector representation of the Lorentz group

During the lecture we have introduced the $(j_- = 1/2, j_+ = 1/2)$ representation. This is a 4 dimensional complex representation that contains a singlet and triplet states, i.e. $\mathbf{1/2} \otimes \mathbf{1/2} = \mathbf{0} \oplus \mathbf{1}$. By analysing the behaviour of a tetravector under rotation we have deduced that this is the (real) vector representation, i.e.

$$\exp\left(-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}\right)^{\alpha}_{\ \beta} = \Lambda^{\alpha}_{\ \beta}\,.$$

In this exercise we will prove it explicitly. Any element of the $(1/2, 1/2)$ representation can be written as (ψ_L, ϕ_R) where ψ_L and ϕ_R are a left-handed and a right handed Weyl spinors, respectively. After having defined

$$\sigma^\mu = (\mathbb{1}, \sigma^i) \qquad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$$

prove that

$$\phi_R^\dagger \sigma^\mu \psi_R \quad \text{and} \quad \phi_L^\dagger \sigma^\mu \psi_L$$

behave as contravariant vectors under Lorentz transformations. Recall the charge conjugate definitions: $\psi_R = i\sigma^2\psi_L^*$ and $\phi_L = -i\sigma^2\phi_R^*$. Additional note: since Λ is a real matrix, one can finally impose the reality conditions to all tetravectors, i.e. $V^\mu = (V^\mu)^*$ Hint: a possible solution is to show that the 4 components transform as a 4-vector under Lorentz transformation. One can, for example, consider a boost along a certain spatial direction and implement the Lorentz transformations on the different Weyl spinors.

2 The Casimir operators of the Poincare‘ group

During the lecture we have stated that the Poincare‘ group has two Casimir operators,P^2 and W^2, where P^μ is the tetramomentum and W^μ is the Pauli-Lubanski tetravector:

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma$$

Using the Lie algebra of the Lorentz group, prove explicitly that these two operators commute with all the generators of the group.

3 The field representation of the tetramomentum

All fields behave as a scalar under spacetime translations. If ψ denote a generic field component (i.e. a scalar field or one of the components of a multicomponent field), then

$$\begin{aligned} x \to x' &= x + a \\ \psi(x) \to \psi(x') &= \psi(x) \end{aligned} \tag{1}$$

Show that the expression of the generator P^μ on the field representation, i.e. on that representation that has the infinite-dimensional space of field function $\{\psi(P)\}_{P \in V^4}$ as base space, is:

$$P^\mu = i\partial^\mu \, .$$

To do that, one has to consider the variation of the field assuming the coordinates to stay the same while the field changes, i.e.

$$(\delta\psi)_0 \equiv \psi'(x) - \psi(x) \, .$$

Hint: consider infinitesimal transformations.