Exercise set 2 for QFT1

November 5, 2019

1 QM recap: the harmonic oscillator

The harmonic oscillator is one of the fundamental QM systems and plays a key role also in QFT. This exercise provides a recap of some fundamental aspects of it. Consider a one-dimensional system described by a state vector $|\psi\rangle$ belonging to an Hilbert space $\mathcal{H}$ and whose Hamiltonian is given (in the Schroedinger picture) by

$$\hat{H}_{\rm S} = \frac{\hat{p}_{\rm S}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}_{\rm S}^2\,,$$

where $\hat{p}_{\rm S}$ and $\hat{q}_{\rm S}$ are the impulse and the displacement operators such that

$$[\hat{q}_{\rm S}, \hat{p}_{\rm S}] = i\,.$$

If

$$\hat{a}_{\rm S} \equiv \sqrt{\frac{m\omega}{2}}\left(\hat{q}_{\rm S} + \frac{i}{m\omega}\hat{p}_{\rm S}\right) \qquad \text{and} \qquad \hat{N}_{\rm S} = \hat{a}_{\rm S}^\dagger \hat{a}_{\rm S}$$

show that:

- $\hat{H}_{\rm S}$ and $\hat{N}_{\rm S}$ share a common basis of eigenstates $\{|n_i\rangle\}$ (here labeled by $\hat{N}_{\rm S}$ eigenvalue) of the Hilbert space $\mathcal{H}$ and find the relation between their eigenvalues;
- $n_i \geq 0$;
- for properly normalized eigenstates,

$$\hat{a}_{\rm S}|n_i\rangle = \sqrt{n_i}\ |n_i - 1\rangle$$
$$\hat{a}_{\rm S}^\dagger|n_i\rangle = \sqrt{n_i + 1}\ |n_i + 1\rangle$$

Given the above properties, n_i must be integer and there must be a ground state $|0\rangle$ such that $a_{\rm S}|0\rangle = 0$. Express the whole basis of $\mathcal{H}$ in terms of $|0\rangle$ and $a_{\rm S}^\dagger$ and compute the matrix elements $\langle m|\hat{q}_{\rm S}|n\rangle$ and $\langle m|\hat{p}_{\rm S}|n\rangle$. Finally compute the temporal evolution in the Heisenberg picture, i.e. find the expressions of $\hat{q}_{\rm H}(t)$ and $\hat{q}_{\rm H}(t)$ in terms of $\hat{q}_{\rm S}$ and $\hat{q}_{\rm S}$ by solving the Heisenberg equations.

2 The Lorentz group

Consider the set of Poincaré transformations over the spacetime $\mathcal{G} = \{T(\Lambda, a)\}$ with Λ a Lorentz matrix and a a translation tetravector. Show that $\mathcal{G}$ is a group. In doing so, find the composition rule for

$$T(\Lambda_2, a_2)\, T(\Lambda_1, a_1)\,,$$

show that $\Lambda_2\Lambda_1$ is a Lorentz matrix, and find the expression for the inverse element. Set $a = 0$ (i.e. consider the homogeneous Lorentz group). Starting from the Lie algebra deduced in the lecture

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho})$$

and from the definitions of $\mathbf{J}$, $\mathbf{K}$, $\mathbf{J}^+$, $\mathbf{J}^-$:

$$\begin{aligned} J^i &\equiv \frac{1}{2}\epsilon^{ijk} J^{jk} \\ K^i &\equiv J^{i0} \\ \mathbf{J}^+ &\equiv \frac{\mathbf{J} + i\mathbf{K}}{2} \\ \mathbf{J}^- &\equiv \frac{\mathbf{J} - i\mathbf{K}}{2} \end{aligned}$$

proof their commutation relations:

$$\begin{aligned} \left[J^i, J^j\right] &= i\epsilon^{ijk} J^k \\ \left[J^i, K^j\right] &= i\epsilon^{ijk} K^k \\ \left[K^i, K^j\right] &= -i\epsilon^{ijk} J^k \end{aligned}$$

and

$$\begin{aligned} \left[J^{+,i}, J^{+,j}\right] &= i\epsilon^{ijk} J^{+,k} \\ \left[J^{-,i}, J^{-,j}\right] &= i\epsilon^{ijk} J^{-,k} \\ \left[J^{+,i}, J^{-,j}\right] &= \mathbb{0} \end{aligned}$$