Exercise set 1 for QFT1

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1 Natural units

In standard unit systems (e.g. SI or cgs) the fundamental dimensions are mass, length and time (M, L and T). In QFT one chooses mass, action and speed (M, A and V). Prove the dimensional identity between the two systems:

$$M^p L^q T^r = M^{p-q-r} A^{q+r} V^{-q-2r} \,. \tag{1}$$

With the latter dimension choice, natural units (n.u.) are easy to adopt. In n.u. the two fundamental constants $\hbar$ and c are set to 1, $\hbar = c = 1$. Thus, a quantity in n.u. has always a dimension M^n. Due to the fact that [energy] $= M$, mass dimensions are often expressed in MeV. Given the expression of a physical quantity in n.u., the conversion between n.u. and standard units is done simply by finding conversion factors for $\hbar$ and c, and by inserting the right powers of $\hbar$ and c as implied by Eq. 1. For example, the Thomson cross-section for a low energy photon scattering off a charged particle is, in n.u.:

$$\sigma = \frac{8\pi}{3} \frac{\alpha^2}{m^2} \,,$$

where $\alpha \approx 1/137$ is the e.m. structure constant and m is the target particle mass. Compute the value of the Thomson cross section for the electron, for which $m = 0.511$ MeV, in cm^2 and in barn (1b $= 10^{-24}$cm^2).
Moreover, the positronium (a bound state of a positron and an electron) has in n.u. a lifetime of:

$$\tau = \frac{2}{\alpha^5} \frac{1}{m}$$

Express it in seconds.
Finally, consider a particle of mass m in its rest frame. Since in natural units

$$[\text{length}] = [\text{energy}]^{-1}$$

then it is natural to introduce a fundamental length scale

$$r_C = \frac{1}{m} \,.$$

Introduce the corrects powers of $\hbar$ and c to compute r_C in non-natural units (e.g. in fm or in cm). Which quantity is thus r_C? Compute r_C for an electron and express it both in MeV and in fm. (Hint: the conversion factors for $\hbar$ and c are easily obtained by computing their numerical values in appropriate units. For example, try to compute $\hbar c$ in MeV fm or in MeV cm).

2 Relativistic kinematics

A particle A, of mass m_A, is sent against another identical particle A, which is initially at rest. In the final state, in addition to the two original particles, there is the creation of a particle-antiparticle pair B-$\bar{B}$, each of mass m_B:

$$A + A \to A + A + B + \bar{B}\,.$$

Find the threshold energy (i.e. the minimum energy of the colliding particle A) for this process to occur as a function of m_A and m_B (hint: consider both the center of mass and lab frame to solve the problem to compute Lorentz invariant quantities. Think about the threshold condition in the center of mass frame rather than in the lab frame). Discuss the results in terms of m_A and m_B, and, in particular, the limits $m_A = m_B$, $m_A \gg m_B$, and $m_A \ll m_B$.

Solve again the excercise by considering the head-on collision of two particle A, i.e. both particles moving at the same speed in the lab frame. To connect the first and the second part of the exercise, let's define the kinetic energy of a particle (in a specific frame) as $T \equiv E - m$. If T is the kinetic energy of each of the two particles in the head-on collision, find the relative kinetic energy T'(defined as the kinetic energy of one in the rest frame of the other). Discuss the limits $T \ll m$ and $T \gg m$.

3 Dirac's δ

Consider a real function of real variable $f(x)$, such that $\{x_{0,i}\}_{i=1,N}$ are the zeros of $f(x)$. Proof that:

$$\delta(f(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_{0,i})}{|f'(x_{0,i})|}$$

where $f'(x)$ is the first derivative of f.
Hint: it could be useful to compute:

$$\int_{-\infty}^{+\infty} \delta(f(x))\, g(x)\, \mathrm{d}x$$

for a generic $g(x)$.